Unified Scaling of Polar Codes: Error Exponent, Scaling Exponent, Moderate Deviations, and Error Floors

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**Existing Results on Scaling of Polar Codes**

<table>
<thead>
<tr>
<th>Error exponent [1, 2]</th>
<th>blue vertical line</th>
<th>What is fixed</th>
<th>How it scales</th>
<th>$P_e \sim 2^{-\sqrt{N}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scaling exponent [3, 4, 5]</td>
<td>red horizontal line</td>
<td>$N \sim 1/(z^* - z)^{\alpha}$</td>
<td>$5.579 \leq \mu \leq 5.702$</td>
<td></td>
</tr>
<tr>
<td>Error floors</td>
<td>green curve</td>
<td>code $(N, R)$</td>
<td>no rigorous result</td>
<td></td>
</tr>
</tbody>
</table>

If $z < z^*$, then $P_e \to 0$ as $N \to \infty$; otherwise, $P_e \to 1$ as $N \to \infty$.

In waterfall region $P_e(z)$ decreases sharply, while in error floor region curves are much more shallow.

**New Results on Scaling of Polar Codes**

<table>
<thead>
<tr>
<th>In figure on top</th>
<th>What is fixed</th>
<th>How it scales</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scaling exponent</td>
<td>red horizontal line</td>
<td>$P_e \leq 4.714$, any $W$</td>
</tr>
<tr>
<td>Moderate deviations</td>
<td>neither $z$ nor $R$</td>
<td>$P_e \sim 2^{-N^{-\frac{1}{10}}}$</td>
</tr>
<tr>
<td>Error floors</td>
<td>green curve</td>
<td>code $(N, R)$</td>
</tr>
</tbody>
</table>

**What’s New in the Bound on $\mu$?**

i) $\mu$ upper bounded by eigenvalue $\mu^*$. Given $h(x)$, we prove that the scaling exponent $\mu$ is upper bounded by any $\mu^*$ s.t.

$$\sup_{x \in [0, 1]} \frac{h(x^2) + h(y)}{2h(x)} < 2^{-1/\nu}.$$  

Idea: $\mu^*$ eigenvalue and $h(x)$ eigenfunction of the operator that evolves the Bhattacharyya process.

ii) Provable bound on $\mu^*$. We devise an algorithm to construct and analyze $h(x)$ in order to give a provable upper bound on the sup in (i).

**Statement of the Bound**

**Theorem.** Assume there exists fixed function $h(x) : [0, 1] \to [0, 1]$ s.t. $h(0) = h(1) = 0$, $h(x) > 0$ for any $x \in (0, 1)$, and (i) holds. Then, $\mu \leq \mu^*$. The value $\mu^* = 4.714$ is achievable.

If $W$ is BEC, a less stringent condition on $\mu^*$ is needed. Instead of (i), we require

$$\sup_{y \in [x^2, 2x^2 - 1]} \frac{h(x^2) + h(y)}{2h(x)} < 2^{-1/\nu}.$$  

The value $\mu^* = 3.639$ is achievable.

**Idea of the proof of $\mu \leq \mu^*$:**

$$\sup_{y \in [x^2, 2x^2 - 1]} \frac{h(x^2) + h(y)}{2h(x)} < 2^{-1/\nu}.$$  

**Interpretation of the Fixed Function - BEC Case**

$$T_{\text{BEC}}(g) = \frac{g(x^2)}{2} + g(2x - z^2)$$

Operator $T_{\text{BEC}}$ linked to the evolution of the Bhattacharyya process $Z_n$,

$$E \{ g(Z_n) \mid Z_0 = z \} = T_{\text{BEC}} \circ T_{\text{BEC}} \circ \cdots \circ T_{\text{BEC}}(g) = T_{\text{BEC}}^n(g).$$

$\lambda = 1$ eigenvalue with eigenfunctions $v_1(z) = 1$ and $v_1(z) = z$,

$$T_{\text{BEC}}(g) \to 0 \cdot 1 + (g(1) - g(0)) \cdot z = (1 - z)g(0) + zg(1).$$

Polarization happens! A fraction $z$ of synthetic channels becomes completely noisy and a fraction $1 - z$ becomes noiseless.

But how fast?

$$T_{\text{BEC}}(g) \to (1 - z)g(0) + zg(1) + (\lambda^*)^2 h(x),$$

Speed of polarization given by second largest eigenvalue $\lambda^* = 2^{-1/\nu}$ with eigenfunction $h(x)$,

$$h(x^2) + h(2x - x^2) < 2^{-1/\nu}.$$  

**References**


