1. Sparse Superposition (SS) Codes

- Original message: \( u \in \{0, 1\}^{L \log_2(2)} \)
- Information word: \( s \in \{0, 1\}^{LB} \) (sparse)
- Coding matrix: \( F \in \mathbb{R}^{M \times N} \), i.i.d. Gaussian entries \( \sim \mathcal{N}(0, 1/L) \)
- Coding rate: \( R = L \log_2(2)/M \)
- Mapping function: \( \pi \) maps continuous-valued codeword to channel-input alphabet.
- \( W \): memoryless channel, e.g. BSC with \( \pi(z) = \text{sign}(z) \).
- Special case: \( W \) is the AWGN channel [1], no map \( \pi \) is needed.

2. Generalized Approximate Message Passing (GAMP)

- Decoder: infer information word \( s \) (sparse) from small number of noisy observations; similar to compressed sensing, up to per-section signal correlation.
  - AWGN channel: adapt AMP [2] to prior distribution of SS codes [3]. Gaussian approximations are applied to reduce the complexity on a complete graph.
  - General memoryless channel: adapt GAMP [4] to SS codes [5,6].
- State evolution [2]: recursion tracking asymptotic MSE \( \{E(i)\} \) of GAMP at each iteration.
  \[ E(i+1) = \text{mse} \left( \Sigma^2(E(i)) \right) \tag{1} \]
- The notion of equivalent AWGN channel is valid, due to the code construction, even if the physical channel is non-Gaussian.
- The physical channel effect is reflected in the equivalent variance \( \Sigma^2(E) \).

3. Asymptotic Performance

- For a fixed channel and alphabet size \( B \), GAMP exhibits sharp phase transition at an algorithmic threshold \( (R_{\text{GAMP}}) \) as \( B \to \infty \) with a gap to capacity. State evolution accurately tracks the performance.
- The existence of error floor \( (E_0) \) at finite \( B \) depends on the channel being used (AWGN: \( E_0 \neq 0 \), binary input channels: \( E_0 = 0 \) [6]).
- As \( B \to \infty \) the error floor, when exists, vanishes. However, the gap to capacity persists.

4. Potential Method - Relation to Statistical Physics

- Potential function is the average “Bethe free energy” \( F_B(E) \) of the graphical model in the thermodynamic limit.
- The stationary points of \( F_B(E) \) correspond to fixed points of \( (1) \).
- The potential (Bayes-optimal) threshold \( R_{\text{pot}} \) is induced from \( F_B(E) \) (the rate at which the two minima occurs at the same energy level).

5. Spatial Coupling - Boost Algorithmic Performance

- Spatial coupling: two ingredients
  - Spatially coupled (SC) construction: spatial dimension to propagate the wave.
  - Perfect side information (seed): initiate the wave.
- Threshold saturation: the algorithmic threshold of the SC code “saturates” the potential threshold in a proper limit: \( R_{\text{GAMP}} = R_{\text{pot}} \).

6. Capacity Achieving Codes at Infinite Alphabet Size

- As \( B \to \infty \) one can get the corresponding potential threshold by equating the two minima of \( F_B(E) \)
  \[ \lim_{B \to \infty} R_{\text{pot}} \approx R_{\text{pot}}^\infty = -\int d\gamma \alpha \gamma \log Z(\gamma) + \int d\gamma \alpha \gamma \log \bigg( W(\gamma) \bigg) \tag{2} \]
- Choose \( \pi \) such that \( \mathcal{P} \) is the capacity achieving input distribution of the channel \( W \), hence \( R_{\text{pot}}^\infty = C \).

7. Practical Codes with Hadamard-Based Operators

- For practical implementations, Hadamard-based operators can be used for the coding matrix:
  - Lower decoding complexity.
  - Less memory needs.
  - Performance cost w.r.t. Gaussian matrices (vanishes as \( B \) increases).
  - Lack of rigor in state evolution analysis.

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